

# A bracket polynomial for graphs

L. Traldi and L. Zulli

**ABSTRACT.** A knot diagram has an associated *looped interlacement graph*, obtained from the intersection graph of the Gauss diagram by attaching loops to the vertices that correspond to negative crossings. This construction suggests an extension of the Kauffman bracket to an invariant of looped graphs, and an extension of Reidemeister equivalence to an equivalence relation on looped graphs. The graph bracket polynomial can be defined recursively using the same pivot and local complementation operations used to define the interlace polynomial, and it gives rise to a graph Jones polynomial  $V_G(t)$  that is invariant under the graph Reidemeister moves.

## 1. Introduction

Shortly after Jones introduced his polynomial invariant of knots in [12], Kauffman described the Jones polynomial in the following way [15]. First, a three-variable bracket polynomial for link diagrams is defined, using either a state sum or a recursion. Then the variables that appear in the bracket polynomial are evaluated so that the result is invariant under Reidemeister moves of the second and third types. Finally, this simplified bracket polynomial is multiplied by an appropriate factor so that the resulting product is invariant under all three types of Reidemeister moves. Thistlethwaite [21] observed that the state sum and recursion used for Kauffman's bracket imply that there is a strong connection between the bracket polynomial of a link diagram and the Tutte polynomial of a graph associated with a checkerboard coloring of the diagram's complementary regions.

In this paper we present a fundamentally different graph-theoretic approach to knot-theoretic ideas related to the Jones polynomial. We take a moment to outline this approach before beginning a detailed presentation.

First we introduce a new graph invariant, the *graph bracket polynomial*. If  $G$  is an  $n$ -vertex graph then  $[G]$  can be described in two different ways. One description is a sum indexed by the  $n \times n$  diagonal matrices over  $GF(2)$ ; this sum is a direct extension of a formula for the Jones polynomial of a classical knot given in [22]. The other description is a recursion involving the local complementation and pivot operations of [1, 2, 3, 7]. The interlace polynomials of [1, 2, 3] can

---

1991 *Mathematics Subject Classification*. Primary 57M25, 05C50.

*Key words and phrases*. graph, knot, Reidemeister move, Jones polynomial, trip matrix, interlace polynomial.

This paper is in final form and no version of it will be submitted for publication elsewhere.

also be defined using either sums of  $2^n$  terms involving  $n \times n$  matrices over  $GF(2)$  or recursions involving the local complementation and pivoting operations, but the graph bracket and interlace polynomials seem to be genuinely different both in definition and in significance. For instance, Sections 5 and 6 of [3] suggest that the interlace polynomials capture more information about the independent sets of a graph than the graph bracket does; on the other hand the graph bracket captures more knot-theoretic information than the interlace polynomials do (see [4]).

Just as the Kauffman bracket  $[D]$  of a knot diagram is related to the Tutte polynomial through the checkerboard graph,  $[D]$  is related to the graph bracket polynomial through a different construction, a looped version of the intersection graph of the Gauss diagram. The concise term *Gauss graph* is already in use, so we refer to this graph as the *looped interlacement graph*,  $\mathcal{L}(D)$ .

DEFINITION 1. *If  $D$  is a plane diagram of a classical (or virtual) knot then the looped interlacement graph  $\mathcal{L}(D)$  has a vertex for each (classical) crossing in  $D$ . Two distinct vertices are adjacent in  $\mathcal{L}(D)$  if and only if the corresponding crossings are interlaced in  $D$ , i.e., while tracing  $D$  (in either direction) one encounters first one crossing, then the other, then the first again, and finally the second again.  $\mathcal{L}(D)$  has a loop at each vertex corresponding to a negative crossing of  $D$ .*

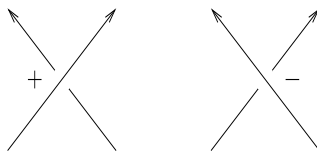


FIGURE 1. positive and negative crossings

$\mathcal{L}(D)$  appeared implicitly in early research regarding the Jones polynomial. Lannes [17] showed that the Arf invariant of a classical knot can be obtained from a combinatorial structure that is essentially a loopless version of  $\mathcal{L}(D)$ ; the relationship between the Arf invariant and the evaluation  $V_K(i)$  was deduced from this result in [12]. Zulli [22] showed that Kauffman's state sum formula for the Jones polynomial of a classical knot can be obtained from the adjacency matrix of  $\mathcal{L}(D)$  (the *trip matrix* of  $D$ ), but did not consider the graph  $\mathcal{L}(D)$ ; the Jones polynomial of a virtual knot can be obtained from  $\mathcal{L}(D)$  in the same way.

In Section 6 we observe that through the looped interlacement graph construction, the Reidemeister moves correspond to a family of simple graph-theoretic operations. Consideration of small examples leads one to guess that a graph Jones polynomial invariant under the graph Reidemeister moves can be obtained from the graph bracket, just as for knots; this guess is verified using the graph bracket's recursive description. As  $\mathcal{L}(D)$  is defined for virtual knot diagrams, these notions share with virtual knot theory some striking differences from the classical case. A fundamental difference is the fact that loop-attachment and -removal (the graph operations that correspond to crossing switches) do not suffice to reduce an arbitrary graph to a Reidemeister equivalent of an edgeless graph; consequently the graph Jones polynomial cannot be computed simply by changing loops and performing Reidemeister moves in order to produce “ungraphs.”

We close this introduction with the observation that these results suggest many questions. How is the graph bracket polynomial of a graph  $G$  related to the interlace polynomials of  $G$ ? Are there graph invariants different from the interlace and graph bracket polynomials that are determined by other recursions involving the pivot and local complementation operations? Which invariants of (classical or virtual) knots extend to Reidemeister equivalence invariants of graphs? Is every graph Reidemeister equivalent to the looped interlacement graph of some virtual knot diagram? Can these notions be extended to links of more than one component?

## 2. Defining the graph bracket polynomial

Definition 1 simply specifies which vertex-pairs are to be adjacent in  $\mathcal{L}(D)$ , and which vertices are to carry loops; multiple edges do not occur in looped interlacement graphs, and modifying the definition to allow multiple edges would not make looped interlacement graphs more useful. We assume similarly that all the graphs we consider in this paper have no multiple edges. The reader who is a stickler for generality may prefer to consider arbitrary (multi-)graphs, with the understanding that every set of multiple edges is to be grouped together and treated as a unit.

DEFINITION 2. *Let  $G$  be a finite graph with vertex-set  $V(G) = \{v_1, \dots, v_n\}$ . The Boolean adjacency matrix  $\mathcal{A}(G) = (a_{ij})$  is the  $n \times n$  matrix over  $GF(2)$  whose  $ij$  entry is 1 or 0 according to whether or not  $G$  has an edge  $\{v_i, v_j\}$ .*

DEFINITION 3. *The graph bracket polynomial of a finite graph  $G$  is*

$$[G](A, B, d) = \sum_{\Delta} A^{\nu(\Delta)} B^{\rho(\Delta)} d^{\nu(\mathcal{A}(G) + \Delta)}$$

*with a summand for each  $n \times n$  diagonal matrix  $\Delta$  over  $GF(2)$ ; here  $\nu$  and  $\rho$  denote the nullity and rank of matrices over  $GF(2)$ , respectively.*

Observe that  $[G]$  is an isomorphism invariant, i.e., it is not affected by the ordering of  $V(G)$ . Note also that no information in  $[G]$  is lost if we replace  $B$  by  $A^{-1}$  or 1; we include  $B$  in Definition 3 only to agree with Kauffman's original bracket notation [15].

Definition 3 is motivated by the observation that if  $D$  is a diagram of a classical knot  $K$  then  $\mathcal{A}(\mathcal{L}(D))$  is the *trip matrix* of  $D$  and hence Theorem 2 of [22] implies that the Kauffman bracket  $[D]$  equals the graph bracket  $[\mathcal{L}(D)]$ . It follows that if  $\mathcal{L}(D)$  has  $n$  vertices and  $\ell$  loops then the Jones polynomial  $V_K(t)$  is the image of  $(-A^3)^\ell (-B^3)^{n-\ell} [\mathcal{L}(D)]$  under the evaluations  $A \mapsto t^{-1/4}$ ,  $B \mapsto t^{1/4}$  and  $d \mapsto -t^{-1/2} - t^{1/2}$ . The computational intractability of  $V_K(t)$  [11] implies that calculating the graph bracket polynomial is also intractable in general.

Definition 3 bears some resemblance to the definition of the two-variable interlace polynomial  $q(G)$  [3]:

$$q(G) = \sum_{S \subseteq V(G)} (x-1)^{\rho(\mathcal{A}(G[S]))} (y-1)^{\nu(\mathcal{A}(G[S]))}$$

where  $G[S]$  is the subgraph of  $G$  induced by  $S$ . Note that  $q(G)$  can be obtained from the sum

$$\sum_{S \subseteq V(G)} [G[S]]$$

by evaluating  $A \mapsto x - 1$ ,  $B \mapsto 0$ , and  $d \mapsto (y - 1)/(x - 1)$ ; the evaluation  $B \mapsto 0$  eliminates most of the terms in the sum, so this is certainly not an efficient way to obtain  $q(G)$ . There is an even more inefficient way to obtain  $[G]$  from a sum involving interlace polynomials. For an  $n \times n$  diagonal matrix  $\Delta$  over  $GF(2)$  let  $G + \Delta$  denote the graph obtained from  $G$  by toggling the loops at the vertices corresponding to nonzero entries of  $\Delta$ , so that  $G + \Delta$  has a loop at a vertex  $v_i$  if and only if either  $G$  has a loop at  $v_i$  and the  $i^{th}$  entry of  $\Delta$  is 0, or  $G$  does not have a loop at  $v_i$  and the  $i^{th}$  entry of  $\Delta$  is 1.  $[G]$  can then be obtained from

$$\sum_{\Delta} z^n A^{\nu(\Delta)} B^{\rho(\Delta)} q(G + \Delta)$$

by first evaluating  $x \mapsto z^{-1} + 1$ ,  $y \mapsto dz^{-1} + 1$  and then evaluating  $z \mapsto 0$ .

In addition to Definition 3, the graph bracket has a recursive definition which involves the local complementation and pivoting operations used by Bouchet [7] and Arratia, Bollobás and Sorkin [1, 2, 3].

**DEFINITION 4.** (*Local Complementation*) *Let  $G$  be a finite graph. If  $a$  is a vertex of  $G$  then  $G^a$  is obtained from  $G$  by toggling adjacencies  $\{x, y\}$  involving neighbors of  $a$  that are distinct from  $a$ . That is, if  $x \neq a \neq y$  and  $x, y$  are neighbors of  $a$  in  $G$  then  $G^a$  contains an edge  $\{x, y\}$  if and only if  $G$  does not.*

Note that the definition allows for the possibility that  $x = y$ , in which case the edge  $\{x, y\}$  is a loop. We will not usually refer to the entire graph  $G^a$ , but rather the subgraph  $G^a - a$  obtained from  $G^a$  by removing  $a$  and all edges incident on  $a$ .

**DEFINITION 5.** (*Pivot*) *Let  $G$  be a finite graph with distinct vertices  $a$  and  $b$ . Then the graph  $G^{ab}$  is obtained from  $G$  by toggling adjacencies  $\{x, y\}$  such that  $x, y \notin \{a, b\}$ ,  $x$  is adjacent to  $a$  in  $G$ ,  $y$  is adjacent to  $b$  in  $G$ , and either  $x$  is not adjacent to  $b$  or  $y$  is not adjacent to  $a$ . That is,  $G^{ab}$  contains such an edge  $\{x, y\}$  if and only if  $G$  does not.*

The recursive definition of the graph bracket is given in Theorem 1. As we will see in Section 4 below, part (i) is an extension to the graph bracket of the switching formula of the Kauffman bracket [14] and the braid-plat formula of the Jones polynomial [5].

**THEOREM 1.** (i) *If  $G$  is a finite graph with a loop at  $a$  then*

$$[G] = A^{-1}B[G - \{a, a\}] + (A - A^{-1}B^2)[G^a - a],$$

*where  $G - \{a, a\}$  is obtained from  $G$  by removing the loop at  $a$ .*

(ii) *If  $a$  and  $b$  are distinct loopless neighbors in  $G$  then*

$$[G] = A^2[G^{ab} - a - b] + AB[(G^{ab})^a - a - b] + B[G^a - a].$$

(iii) *The empty graph  $E_0$  has  $[E_0] = 1$ , and the edgeless graph  $E_n$  with  $n \geq 1$  vertices has  $[E_n] = (Ad + B)^n$ .*

If  $G$  is an  $n$ -vertex graph then a recursive calculation of  $[G]$  using Theorem 1 yields a formula  $[G] = \sum_{i=0}^n c_i [E_i]$  for some coefficients  $c_0, \dots, c_n$  which are integer polynomials in  $A$ ,  $A^{-1}$  and  $B$ . These coefficients are uniquely determined:  $c_n$  is the coefficient of  $d^n$  in  $A^{-n}[G]$ ,  $c_{n-1}$  is the coefficient of  $d^{n-1}$  in  $A^{1-n}[G] - A^{1-n}c_n(Ad + B)^n$ , and so on. It follows that the graph bracket polynomial is “universal” among graph invariants that satisfy parts (i) and (ii) of Theorem 1, i.e., any such graph

invariant is determined by the graph bracket. For instance, suppose  $A, B, X_0, X_1, \dots$  are independent indeterminates, with  $A$  invertible; then there is a graph invariant  $[[G]]$  that satisfies parts (i) and (ii) of Theorem 1 and has  $[[E_n]] = X_n$  for every  $n$ . One might expect this invariant to be more sensitive than the graph bracket, because it involves infinitely many indeterminates. Instead it contains precisely the same information as the graph bracket does: for every graph  $G$ ,  $[[G]] = \sum_{i=0}^{|V(G)|} c_i X_i$  and  $[G] = \sum_{i=0}^{|V(G)|} c_i [E_i]$ , with the same coefficients  $c_i$ .

We proceed to prove Theorem 1. Observe first that  $[G]$  is unchanged if we permute the vertices of  $G$ , so we may always presume that the vertices of  $G$  are ordered in a convenient way.

Suppose  $G$  has a loop at  $a = v_1$ , and let

$$\begin{aligned} \sum_0 &= \sum_{\Delta=(\delta_{ij}) \text{ with } \delta_{11}=0} A^{\nu(\Delta)} B^{\rho(\Delta)} d^{\nu(\mathcal{A}(G)+\Delta)} \\ \text{and } \sum_1 &= \sum_{\Delta=(\delta_{ij}) \text{ with } \delta_{11}=1} A^{\nu(\Delta)} B^{\rho(\Delta)} d^{\nu(\mathcal{A}(G)+\Delta)}. \end{aligned}$$

Then  $[G] = \sum_0 + \sum_1$  and  $[G - \{a, a\}] = A^{-1}B \sum_0 + AB^{-1} \sum_1$ , so

$$[G] = A^{-1}B[G - \{a, a\}] + (1 - A^{-2}B^2) \sum_0.$$

Suppose  $\Delta = (\delta_{ij})$  is a diagonal matrix with  $\delta_{11} = 0$ . Then

$$\mathcal{A}(G) + \Delta = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & M_{11} & M_{12} \\ \mathbf{0} & M_{21} & M_{22} \end{pmatrix},$$

where bold characters indicate row and column vectors with all entries the same. If  $M_{11}^c$  denotes the matrix obtained by changing every entry in  $M_{11}$  then

$$\nu(\mathcal{A}(G) + \Delta) = \nu \begin{pmatrix} M_{11}^c & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \nu(\mathcal{A}(G^a - a) + \Delta'),$$

where  $\Delta'$  is the submatrix of  $\Delta$  obtained by removing the first row and column. Consequently

$$\begin{aligned} \sum_0 &= \sum_{n \times n \Delta \text{ with } \delta_{11}=0} A^{\nu(\Delta)} B^{\rho(\Delta)} d^{\nu(\mathcal{A}(G)+\Delta)} \\ &= \sum_{(n-1) \times (n-1) \Delta'} A^{1+\nu(\Delta')} B^{\rho(\Delta')} d^{\nu(\mathcal{A}(G^a-a)+\Delta')} = A[G^a - a]. \end{aligned}$$

This completes the proof of part (i) of Theorem 1. Suppose  $a = v_1$  and  $b = v_2$  are adjacent in  $G$ , and  $G$  does not have a loop at  $a$  or  $b$ . For  $\beta \in \{0, 1\}$  let

$$\sum_{0,\beta} = \sum_{\Delta=(\delta_{ij}) \text{ with } \delta_{11}=0 \text{ and } \delta_{22}=\beta} A^{\nu(\Delta)} B^{\rho(\Delta)} d^{\nu(\mathcal{A}(G)+\Delta)}.$$

Let  $\Delta$  be a diagonal matrix with  $\delta_{11} = 0 = \delta_{22}$ . Then

$$\mathcal{A}(G) + \Delta = \begin{pmatrix} 0 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & M_{11} & M_{12} & M_{13} & M_{14} \\ \mathbf{1} & \mathbf{0} & M_{21} & M_{22} & M_{23} & M_{24} \\ \mathbf{0} & \mathbf{1} & M_{31} & M_{32} & M_{33} & M_{34} \\ \mathbf{0} & \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}.$$

Using row and column operations to eliminate the  $\mathbf{1}$  vectors, we see that the nullity of  $\mathcal{A}(G) + \Delta$  is the same as that of

$$\begin{pmatrix} 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & M_{11}^c & M_{12}^c & M_{13} & M_{14} \\ \mathbf{1} & \mathbf{0} & M_{21} & M_{22} & M_{23} & M_{24} \\ \mathbf{0} & \mathbf{0} & M_{31}^c & M_{32}^c & M_{33} & M_{34} \\ \mathbf{0} & \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_{11} & M_{12}^c & M_{13}^c & M_{14} \\ \mathbf{0} & \mathbf{0} & M_{21}^c & M_{22} & M_{23}^c & M_{24} \\ \mathbf{0} & \mathbf{0} & M_{31}^c & M_{32}^c & M_{33} & M_{34} \\ \mathbf{0} & \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}.$$

Observe that

$$\begin{pmatrix} M_{11} & M_{12}^c & M_{13}^c & M_{14} \\ M_{21}^c & M_{22} & M_{23}^c & M_{24} \\ M_{31}^c & M_{32}^c & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} = \mathcal{A}(G^{ab} - a - b) + \Delta',$$

where  $\Delta'$  is obtained from  $\Delta$  by removing the first two rows and columns. As  $\rho(\Delta) = \rho(\Delta')$  and  $\nu(\Delta) = \nu(\Delta') + 2$ , summing over all such  $\Delta$  tells us that

$$\sum_{0,0} = A^2[G^{ab} - a - b].$$

If  $\Delta$  is a diagonal matrix with  $\delta_{11} = 0 \neq \delta_{22}$  then the nullity of

$$\mathcal{A}(G) + \Delta = \begin{pmatrix} 0 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & M_{11} & M_{12} & M_{13} & M_{14} \\ \mathbf{1} & \mathbf{0} & M_{21} & M_{22} & M_{23} & M_{24} \\ \mathbf{0} & \mathbf{1} & M_{31} & M_{32} & M_{33} & M_{34} \\ \mathbf{0} & \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$

is the same as the nullity of

$$\begin{pmatrix} 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & M_{11}^c & M_{12}^c & M_{13} & M_{14} \\ \mathbf{1} & \mathbf{0} & M_{21} & M_{22} & M_{23} & M_{24} \\ \mathbf{0} & \mathbf{0} & M_{31}^c & M_{32}^c & M_{33} & M_{34} \\ \mathbf{0} & \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_{11} & M_{12} & M_{13}^c & M_{14} \\ \mathbf{0} & \mathbf{0} & M_{21} & M_{22} & M_{23}^c & M_{24} \\ \mathbf{0} & \mathbf{0} & M_{31}^c & M_{32}^c & M_{33} & M_{34} \\ \mathbf{0} & \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}.$$

If  $\Delta'$  is obtained from  $\Delta$  by removing the first two rows and columns then

$$\begin{pmatrix} M_{11}^c & M_{12} & M_{13}^c & M_{14} \\ M_{21}^c & M_{22} & M_{23}^c & M_{24} \\ M_{31}^c & M_{32}^c & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} = \mathcal{A}((G^{ab})^a - a - b) + \Delta'.$$

As  $\rho(\Delta) = \rho(\Delta') + 1$  and  $\nu(\Delta) = \nu(\Delta') + 1$ , summing over all such  $\Delta$  shows that

$$\sum_{0,1} = AB[(G^{ab})^a - a - b].$$

Let

$$\sum_1 = \sum_{\Delta=(\delta_{ij}) \text{ with } \delta_{11}=1} A^{\nu(\Delta)} B^{\rho(\Delta)} d^{\nu(\mathcal{A}(G)+\Delta)}.$$

If  $\Delta$  is a diagonal matrix with  $\delta_{11} = 1$  then the nullity of

$$\mathcal{A}(G) + \Delta = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & M_{11} & M_{12} \\ \mathbf{0} & M_{21} & M_{22} \end{pmatrix}$$

is the same as the nullity of

$$\begin{pmatrix} M_{11}^c & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \mathcal{A}(G^a - a) + \Delta',$$

where  $\Delta'$  is obtained from  $\Delta$  by removing the first row and column. As  $\rho(\Delta) = \rho(\Delta') + 1$  and  $\nu(\Delta) = \nu(\Delta')$ , summing over all such  $\Delta$  tells us that

$$\sum_1 = B[G^a - a].$$

The above formulas for  $\sum_{0,0}$ ,  $\sum_{0,1}$  and  $\sum_1$  imply part (ii) of Theorem 1.

Part (iii) follows immediately from Definition 3.  $\square$

### 3. Some properties of the graph bracket

PROPOSITION 1. *If  $G$  is the union of disjoint subgraphs  $G_1$  and  $G_2$  then  $[G] = [G_1] \cdot [G_2]$ .*

PROOF. Each matrix  $\mathcal{A}(G) + \Delta$  which appears in Definition 3 consists of two diagonal blocks  $\mathcal{A}(G_1) + \Delta_1$  and  $\mathcal{A}(G_2) + \Delta_2$ .  $\square$

PROPOSITION 2. *Let  $G$  be a graph and let  $G + I$  denote the graph obtained by toggling the loops at the vertices of  $G$  (i.e.,  $G + I$  has loops at precisely those vertices where  $G$  does not). Then  $[G + I](A, B, d) = [G](B, A, d)$ .*

PROOF. Let  $I$  be the  $n \times n$  identity matrix. Then

$$\begin{aligned} [G + I](A, B, d) &= \sum_{\Delta} A^{\nu(\Delta)} B^{\rho(\Delta)} d^{\nu(\mathcal{A}(G+I)+\Delta)} \\ &= \sum_{\Delta} A^{\rho(I+\Delta)} B^{\nu(I+\Delta)} d^{\nu(\mathcal{A}(G)+I+\Delta)} = [G](B, A, d). \end{aligned}$$

This argument is essentially the same as the proof of Proposition 3 of [20].  $\square$

Two simplifications of the graph bracket are defined just as for knots.

DEFINITION 6. *The reduced graph bracket of  $G$  is  $\langle G \rangle(A) = [G](A, A^{-1}, -A^2 - A^{-2})$ . If  $G$  has  $n$  vertices and  $\ell$  loops then the graph Jones polynomial of  $G$  is  $V_G(t) = (-1)^n \cdot t^{(3n-6\ell)/4} \cdot \langle G \rangle(t^{-1/4})$ .*

Some of the properties of Jones polynomials of classical knots that were given in [12] do not extend to the Jones polynomials of arbitrary graphs. For instance  $V_K(t)$  is always a Laurent polynomial in  $t$ , but odd powers of  $t^{1/2}$  may appear in  $V_G(t)$ ; e.g., the 2-vertex path  $P_2$  (a non-loop edge with its two end-vertices) has  $V_{P_2}(t) = t^{3/2}(t^{-1/2} + 1 - t)$ . (N.b. Definition 3 implies that  $\langle G \rangle(A)$  involves only powers  $A^k$  with  $k \equiv n \pmod{2}$ , so  $V_G(t)$  is always a Laurent polynomial in  $t^{1/2}$ .) It follows that results about specific values of  $V_K(t)$  cannot be unambiguously generalized to  $V_G(t)$ . For example, classical knots have  $V_K(e^{2\pi i/3}) = 1 = \pm V_K(i)$ , but  $V_{P_2}(e^{2\pi i/3})$  is  $-2 + i\sqrt{3}$  or 1 according to the choice of  $(e^{2\pi i/3})^{1/2}$ , and similarly  $V_{P_2}(i)$  is  $i(1 \pm \sqrt{2})$ . Every classical knot diagram can be changed into a diagram of the trivial knot by reversing some crossings, and consequently if  $\mathcal{L}(D)$  is the looped interlacement graph of a classical knot diagram then there is a graph  $G$  that differs from  $\mathcal{L}(D)$  only with regard to loops and has  $V_G(t) = 1$ . This is another property that does not extend to the Jones polynomials of arbitrary graphs; for instance, it is not possible to obtain a graph  $G$  with  $V_G(t) = 1$  by adjoining loops to the 6-vertex path  $P_6$ .

By the way, both  $P_2$  and  $P_6$  are looped interlacement graphs of virtual knots, so the “misbehavior” mentioned in the preceding paragraph occurs when the Jones polynomial is extended from classical knots to virtual knots. We do not know if there is a graph whose Jones polynomial is not the Jones polynomial of some virtual knot.

Theorems 15 and 16 of [12], on the other hand, do extend to arbitrary graphs.

**PROPOSITION 3.** *If we use 1 for  $1^{-1/4}$  then every graph has  $V_G(1) = 1$  and  $V'_G(1) = 0$ .*

**PROOF.** The equality  $V_G(1) = 1$  is equivalent to  $\langle G \rangle(1) = (-1)^n$ . This latter is certainly true for  $\langle E_n \rangle = (-A^3)^n$ , and Theorem 1 directly provides a general inductive proof.

Note that

$$\begin{aligned} V'_G(1) &= (-1)^n \left( \frac{3n - 6\ell}{4} \right) \cdot \langle G \rangle(1) + (-1)^n \frac{d}{dt} (\langle G \rangle(t^{-1/4}))(1) \\ &= \left( \frac{3n - 6\ell}{4} \right) + (-1)^n \frac{d}{dt} (\langle G \rangle(t^{-1/4}))(1). \end{aligned}$$

Consequently  $V'_G(1) = 0$  if  $\frac{d}{dt} (\langle G \rangle(t^{-1/4}))(1) = (-1)^n \left( \frac{6\ell - 3n}{4} \right)$ ; the equivalent formula  $\frac{d\langle G \rangle}{dA}(1) = (-1)^n(3n - 6\ell)$  is verified in Proposition 4 below.  $\square$

Note that Proposition 3 and Theorem 15 of [12] together imply that the graph bracket (resp. the Jones polynomial) of a graph cannot equal the Kauffman bracket (resp. the Jones polynomial) of a link of two or more components.

**PROPOSITION 4.** *The graph bracket polynomial of  $G$  determines both the number  $n$  of vertices in  $G$  and the number  $\ell$  of loops in  $G$ :*

$$n = \log_2([G](1, 1, 1)) \quad \text{and} \quad \ell = \frac{n}{2} - \left( \frac{(-1)^n}{6} \right) \cdot \frac{d\langle G \rangle}{dA}(1).$$

**PROOF.** Definition 3 immediately implies that  $[G](A, B, 1) = (A + B)^n$ , and hence  $[G](1, 1, 1) = 2^n$ .



The equality  $\frac{d\langle G \rangle}{dA}(1) = (-1)^n(3n - 6\ell)$  is certainly true for  $\langle E_n \rangle = (-1)^n A^{3n}$ . Theorem 1 provides a general inductive proof as follows.

If  $G$  has a loop at  $a$  then

$$\langle G \rangle = A^{-2} \langle G - \{a, a\} \rangle + (A - A^{-3}) \langle G^a - a \rangle.$$

Using the inductive hypothesis and the equality  $\langle G \rangle(1) = (-1)^n$ , we conclude that

$$\begin{aligned} \frac{d\langle G \rangle}{dA}(1) &= -2 \langle G - \{a, a\} \rangle(1) + (-1)^n(3n - 6(\ell - 1)) + 4 \langle G^a - a \rangle(1) \\ &= (-1)^n \cdot (-2 + 3n - 6\ell + 6 - 4). \end{aligned}$$

Suppose  $G$  has no loops and  $\{a, b\}$  is an edge of  $G$ . Then

$$\langle G \rangle = A^2 \langle G^{ab} - a - b \rangle + \langle (G^{ab})^a - a - b \rangle + A^{-1} \langle G^a - a \rangle.$$

$G^{ab} - a - b$  is loopless,  $G^a - a$  has  $\deg(a)$  loops, and  $(G^{ab})^a - a - b$  has  $\deg(a) - 1$  loops; consequently the inductive hypothesis and the equality  $\langle G \rangle(1) = (-1)^n$  imply

$$\begin{aligned} \frac{d\langle G \rangle}{dA}(1) &= 2(-1)^{n-2} + (-1)^{n-2}(3n - 6) + (-1)^{n-2}(3n - 6 - 6(\deg(a) - 1)) \\ &\quad - (-1)^{n-1} + (-1)^{n-1}(3n - 3 - 6\deg(a)) \\ &= (-1)^n \cdot (2 + 3n - 6 + 3n - 6\deg(a) + 1 - 3n + 3 + 6\deg(a)). \end{aligned}$$

□

#### 4. Applying Theorem 1 to knots

Theorem 1 results in recursive algorithms that can be used to calculate the bracket and Jones polynomials of (virtual) knot diagrams. These algorithms have two distinctive features: links of more than one component never appear, and the recursions involve simple reduction of the crossing number without any reference to “unknotting.” As knots and links are usually treated together in the literature regarding the bracket and Jones polynomials, it may be that these algorithms have not appeared before.

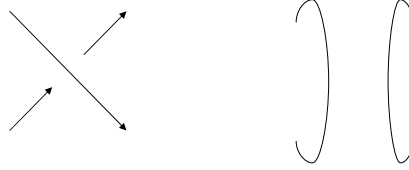


FIGURE 2.  $D_\infty = D_B$  for a positive crossing

Clearly crossing-switches in a knot diagram give rise to loop-toggles in the looped interlacement graph. Moreover, a diagram  $D$  of a knot  $K$  yields a diagram  $D_\infty$  of a knot  $K_\infty$  by splicing together the two arcs directed into a crossing, and also splicing together the two arcs directed out of that crossing, as in Figure 2. The portion of  $K_\infty$  outside the pictured area consists of two arcs; the orientation of  $K_\infty$  along one arc is the same as that of  $K$  while the orientation of  $K_\infty$  along the other arc is the reverse of that of  $K$ . The effect of this partial orientation-reversal on the looped intersection graph is simple: if  $a$  is the vertex of  $\mathcal{L}(D)$  corresponding to the

pictured crossing then  $\mathcal{L}(D_\infty) = \mathcal{L}(D)^a - a$ . Consequently part (i) of Theorem 1 is the extension to the graph bracket polynomial of the Jones polynomial's braid-plat formula ( $tV_{-1} - V_1 = t^{3q}(t-1)V_\infty$  in the notation of [5]) and the Kauffman bracket's switching formula ( $A\chi - A^{-1}\bar{\chi} = (A^2 - A^{-2})\asymp$  in the notation of [14]).

For a positive crossing like the one in Figure 2,  $D_\infty$  is also denoted  $D_B$ . The opposite smoothing, pictured in Figure 3, is denoted  $D_0$  or  $D_A$ . At a crossing that differs from the pictured one with regard to the orientation of  $K$ , the designations of  $D_A$  and  $D_B$  are the same but the designations of  $D_\infty$  and  $D_0$  may be reversed;  $D_0$  always denotes the diagram obtained by smoothing a crossing of  $D$  in a manner consistent with the orientation of  $K$ .



FIGURE 3.  $D_0 = D_A$  for a positive crossing

Suppose  $D$  is an  $n$ -crossing knot diagram with two interlaced positive crossings corresponding to vertices  $a$  and  $b$  of  $\mathcal{L}(D)$ . We denote the  $(n-2)$ -crossing diagrams obtained by smoothing both crossings by first indicating the smoothing at the  $a$  crossing. For instance  $D_{\infty 0} = D_{BB}$  denotes the diagram obtained by smoothing the  $a$  crossing against orientation and then smoothing the  $b$  crossing with orientation; as the two crossings are interlaced the  $b$  crossing becomes negative when the  $a$  crossing is smoothed, so its orientation-consistent smoothing is  $(D_\infty)_B$  rather than  $(D_\infty)_A$ . The fundamental recursive formula for the Kauffman bracket is  $[D] = A[D_A] + B[D_B]$ ; applying this fundamental formula twice yields

$$[D] = A^2[D_{AA}] + AB[D_{AB}] + B[D_B].$$

The extension of this formula to the graph bracket polynomial is part (ii) of Theorem 1. Consequently, the recursive description of the graph bracket polynomial given in Theorem 1 specializes to the following algorithm for the bracket polynomial of a (virtual) knot diagram.

1. If  $D_-$  is a diagram that contains a negative (classical) crossing then apply the 3-variable form of the switching formula of the Kauffman bracket,  $A[D_-] = B[D_+] + (A^2 - B^2)[D_\infty]$ , where  $D_+$  is the diagram obtained by reversing that negative crossing.
2. If  $D$  is a diagram with only positive (classical) crossings, apply the formula  $[D] = A^2[D_{AA}] + AB[D_{AB}] + B[D_B]$  to any interlaced pair of crossings.
3. Repeat steps 1 and 2 as necessary.
4. A diagram with  $n$  crossings, all positive and no two interlaced, has bracket polynomial  $(Ad + B)^n$ .

The resulting algorithm for the Jones polynomial involves formulas that are slightly more complicated, because obtaining the Jones polynomial from the bracket involves not only an evaluation but also multiplication by  $(-1)^n \cdot t^{3w(D)/4}$ . In the

notation of [5], step 1 involves the formula  $tV_{-1} - V_1 = t^{3q}(t-1)V_\infty$  and step 2 involves the formula  $V_{11} = tV_{00} + t^{3q}V_{0\infty} - t^{1+3q}V_\infty$ ; in each formula  $q$  is the linking number of the two components of the link diagrammed in  $D_0$ .

### 5. Examples of graph bracket polynomials

EXAMPLE 1. *Let  $K_n$  be the complete simple graph on  $n$  vertices. Then*

$$[K_n] = \frac{(A+Bd)^n - A^n}{d} + \begin{cases} A^n d, & \text{if } n \text{ is odd} \\ A^n, & \text{if } n \text{ is even} \end{cases}.$$

PROOF. The proposition is true for  $K_0 = E_0$  and  $K_1 = E_1$ .

Proceeding inductively, suppose  $n \geq 2$ . Proposition 1 and part (ii) of Theorem 1 tell us that  $[K_n] = A^2[K_{n-2}] + AB[L_1]^{n-2} + B[L_1]^{n-1}$ , where  $L_1$  is the graph with one vertex and one loop. Consequently

$$\begin{aligned} [K_n] &= A^2[K_{n-2}] + AB(A+Bd)^{n-2} + B(A+Bd)^{n-1} \\ &= A^2[K_{n-2}] + (A+Bd)^{n-2} \cdot (2AB + B^2d) \\ &= \frac{A^2(A+Bd)^{n-2} - A^n}{d} + (A+Bd)^{n-2} \cdot (2AB + B^2d) + \begin{cases} A^n d, & \text{if } n \text{ is odd} \\ A^n, & \text{if } n \text{ is even} \end{cases} \\ &= \frac{(A+Bd)^{n-2}(A^2 + 2ABd + B^2d^2) - A^n}{d} + \begin{cases} A^n d, & \text{if } n \text{ is odd} \\ A^n, & \text{if } n \text{ is even} \end{cases}. \end{aligned}$$

□

EXAMPLE 2. *Let  $P_n$  be the simple path with  $n$  vertices, and let  $L_n$  be the graph obtained from  $P_n$  by adding a loop to one vertex at the end of the path. ( $L_n$  is a lollipop.) Then*

$$[L_n] = (A - A^{-1}B^2)^n + A^{-1}B \sum_{i=0}^{n-1} (A - A^{-1}B^2)^i [P_{n-i}].$$

PROOF. The proposition holds when  $n = 1$ , for  $[L_1] = A + Bd = A - A^{-1}B^2 + A^{-1}B(Ad + B)$ . Proceeding inductively, if  $n > 1$  then part (i) of Theorem 1 tells us that  $[L_n] = A^{-1}B[P_n] + (A - A^{-1}B^2)[L_{n-1}]$ . □

EXAMPLE 3. *Let  $P_n$  be the simple path with  $n$  vertices. Then*

$$\begin{aligned} [P_n] &= \gamma_1(A^{-1}B^2 - A)^n + \gamma_2(A+B)^n \\ &\quad + \gamma_3 2^{-n} \left( -B + \sqrt{4A^2 - 3B^2} \right)^n + \gamma_4 2^{-n} \left( -B - \sqrt{4A^2 - 3B^2} \right)^n \\ \text{with } \gamma_1 &= 0, \quad \gamma_2 = \frac{d+2}{3}, \quad \gamma_3 = \frac{(4A - 3B - \sqrt{4A^2 - 3B^2})(d-1)}{6\sqrt{4A^2 - 3B^2}} \\ \text{and } \gamma_4 &= \frac{(3B - 4A - \sqrt{4A^2 - 3B^2})(d-1)}{6\sqrt{4A^2 - 3B^2}}. \end{aligned}$$

PROOF. Suppose for the moment that  $n \geq 4$ . Example 2 and part (ii) of Theorem 1 tell us that

$$\begin{aligned} [P_n] &= (A^2 + AB)[P_{n-2}] + B[L_{n-1}] \\ &= (A^2 + AB)[P_{n-2}] + B(A - A^{-1}B^2)^{n-1} + A^{-1}B^2 \sum_{i=0}^{n-2} (A - A^{-1}B^2)^i [P_{n-1-i}] \end{aligned}$$

Comparing this with

$$[P_{n-2}] = (A^2 + AB)[P_{n-4}] + B(A - A^{-1}B^2)^{n-3} + A^{-1}B^2 \sum_{i=0}^{n-4} (A - A^{-1}B^2)^i [P_{n-3-i}]$$

we see that

$$\begin{aligned} [P_n] &= (A - A^{-1}B^2)^2 [P_{n-2}] - (A - A^{-1}B^2)^2 (A^2 + AB) [P_{n-4}] \\ &\quad + (A^2 + AB) [P_{n-2}] + A^{-1}B^2 [P_{n-1}] + A^{-1}B^2 (A - A^{-1}B^2) [P_{n-2}] \\ &= A^{-1}B^2 [P_{n-1}] + (A + B)(2A - B) [P_{n-2}] - (A^2 + AB)(A - A^{-1}B^2)^2 [P_{n-4}]. \end{aligned}$$

Consequently the  $[P_n]$  satisfy a linear recurrence with characteristic polynomial

$$\begin{aligned} x^4 - A^{-1}B^2x^3 - (A + B)(2A - B)x^2 + (A^2 + AB)(A - A^{-1}B^2)^2 \\ = (x + A - A^{-1}B^2)(x - A - B)(x^2 + Bx - A^2 + B^2); \end{aligned}$$

this implies that

$$\begin{aligned} [P_n] &= \gamma_1 (A^{-1}B^2 - A)^n + \gamma_2 (A + B)^n \\ &\quad + \gamma_3 2^{-n} \left( -B + \sqrt{4A^2 - 3B^2} \right)^n + \gamma_4 2^{-n} \left( -B - \sqrt{4A^2 - 3B^2} \right)^n \end{aligned}$$

for some coefficients  $\gamma_i$ . The reader may verify that the  $\gamma_i$  given in the statement yield the correct values of  $[P_n]$  for  $0 \leq n \leq 3$ .  $\square$

Here is another version of the formula of Example 3.

$$\begin{aligned} [P_n] &= \left( \frac{d+2}{3} \right) (A + B)^n - \left( \frac{d-1}{3 \cdot 2^n} \right) \begin{cases} (4A^2 - 3B^2)^{n/2}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases} \\ &\quad + \left( \frac{d-1}{3 \cdot 2^{n-2}} \right) A \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (4A^2 - 3B^2)^j (-B)^{n-2j-1} \binom{n}{2j+1} \\ &\quad + \left( \frac{d-1}{3 \cdot 2^n} \right) \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (4A^2 - 3B^2)^j (-B)^{n-2j} \left( 3 \binom{n}{2j+1} - \binom{n}{2j} \right). \end{aligned}$$

Computer calculations indicate that the graph bracket polynomial is surprisingly effective in distinguishing small graphs: it distinguishes all the non-isomorphic graphs with no more than 5 vertices, and also the simple graphs with 6 vertices. (Indeed, the one-variable simplification  $[G](A, 1, A)$  distinguishes all these graphs.) There are 5027 different graph bracket polynomials among the 5096 non-isomorphic 6-vertex looped graphs, and 1028 different graph bracket polynomials among the 1044 non-isomorphic simple 7-vertex graphs.

The graph bracket polynomial distinguishes trees with fewer than 7 vertices, so in general  $[G]$  is not determined by the cycle matroid of  $G$ . Even the graph Jones polynomial distinguishes some trees, despite being a much less sensitive invariant than the graph bracket. It would seem, then, that the graph bracket polynomial provides a novel combinatorial understanding of the Kauffman bracket and the

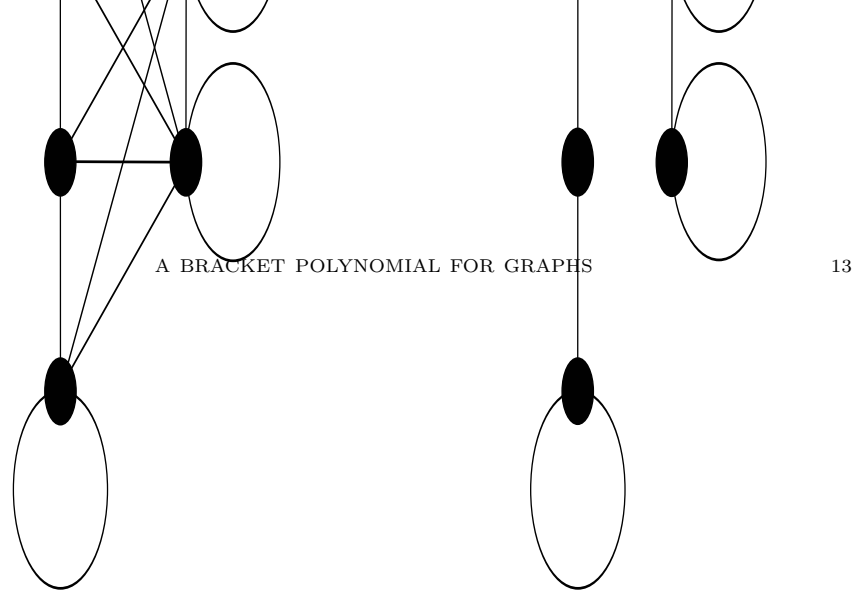


FIGURE 4. two 6-vertex graphs with the same graph bracket polynomial

Jones polynomial, quite different from the very important relationship between these knot invariants and the Tutte polynomial originally observed by Thistlethwaite [21]. (See [10, 11] and their references for examples of results which have been derived using the Jones-Tutte relationship.)

When the graph bracket polynomial does fail to distinguish two graphs, it can be surprisingly insensitive to structural differences between them. For instance, the connected graph on the left in Figure 4 has the same graph bracket as the disconnected graph on the right.

## 6. The knot theory of looped interlacement graphs

The Reidemeister moves [19] give a combinatorial description of the relationship among the different diagrams of a given knot type. The following definition describes the equivalence relation on graphs that is generated by the “images” of these Reidemeister moves under the looped interlacement graph construction.

**DEFINITION 7.** *Two graphs are Reidemeister equivalent if one can be obtained from the other through some finite sequence of the following Reidemeister moves.*

**Ω.1.** *Adjoin or remove an isolated vertex. The isolated vertex may be looped or unlooped.*

**Ω.2.** *Adjoin or remove two vertices  $v$  and  $w$  such that (i)  $v$  is looped, (ii)  $w$  is not looped, and (iii) every vertex  $x \notin \{v, w\}$  that is adjacent to one of  $v, w$  is also adjacent to the other. The two vertices  $v$  and  $w$  may be adjacent or nonadjacent.*

**Ω.3.** *Toggle the non-loop adjacencies among three distinct vertices  $u, v$  and  $w$  such that (i) every vertex  $x \notin \{u, v, w\}$  that is adjacent to any of  $u, v, w$  is adjacent to precisely two of  $u, v, w$  and (ii) either the initial or the terminal subgraph spanned by  $u, v, w$  is isomorphic to one of the three-vertex graphs pictured in Figure 5.*

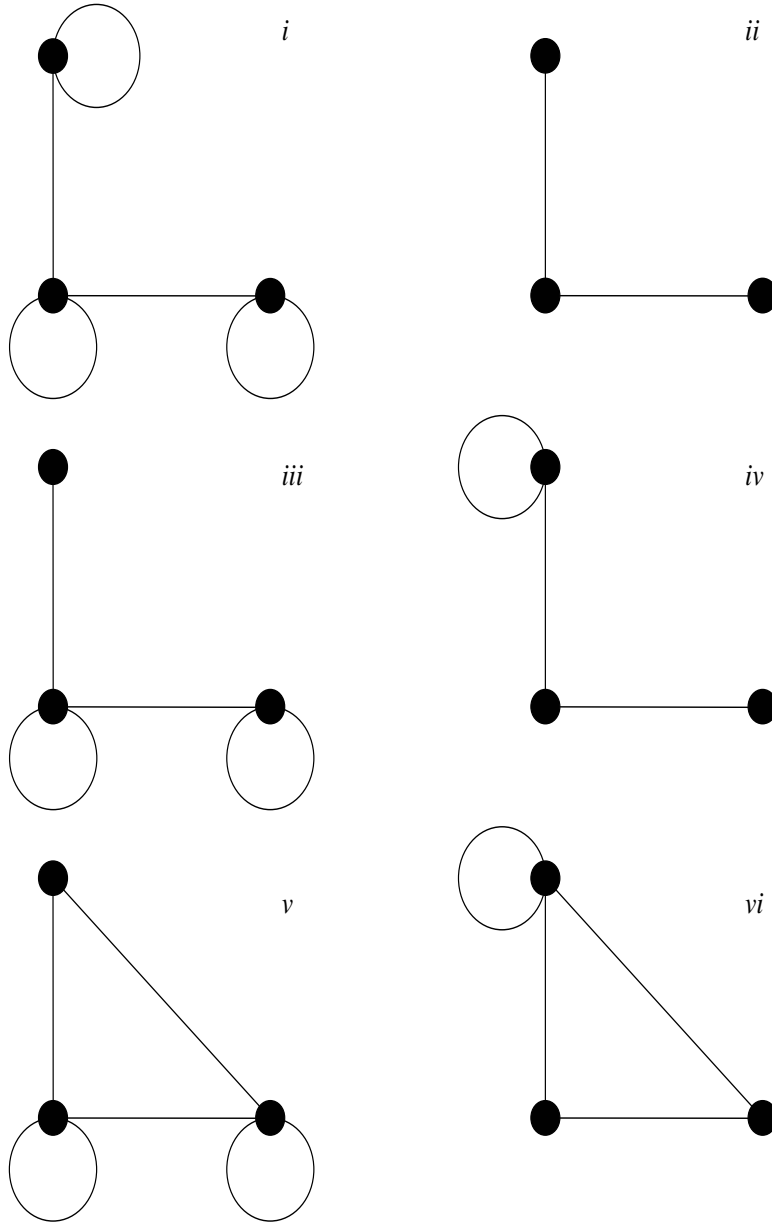


FIGURE 5. An  $\Omega.3$  move involves toggling the non-loop edges in one of these six configurations, provided that every vertex outside the picture is adjacent to 0 or precisely 2 of the pictured vertices.

**THEOREM 2.** *If  $D_1$  and  $D_2$  are diagrams of the same (classical or virtual) knot then the looped interlacement graphs  $\mathcal{L}(D_1)$  and  $\mathcal{L}(D_2)$  are Reidemeister equivalent.*

Let  $\mathcal{D}$  denote the class of virtual (possibly classical) knot diagrams, and let  $\mathcal{G}$  denote the class of looped graphs. Theorem 2 implies that the function  $\mathcal{L}: \mathcal{D} \rightarrow \mathcal{G}$

induces a function  $\tilde{\mathcal{L}}: \mathcal{D}/\sim \rightarrow \mathcal{G}/\sim$ , where  $\sim$  denotes Reidemeister equivalence. That is,  $\tilde{\mathcal{L}}$  is a knot invariant. This function is not injective, since the looped interlacement graph of a knot diagram may be equivalent (even identical) to the looped interlacement graph of a diagram of a different knot. Not all graphs are equivalent to looped interlacement graphs of classical knot diagrams, but we do not know whether or not  $\tilde{\mathcal{L}}$  is surjective.

Since  $\tilde{\mathcal{L}}$  is a knot invariant, so is  $f \circ \tilde{\mathcal{L}}$  for any function  $f: \mathcal{G}/\sim \rightarrow X$ . Such extended invariants are difficult to find because the graph-theoretic Reidemeister moves affect the structure of a graph dramatically, changing the number of vertices, number of edges, degree sequence, connectedness, chromaticity, cycle structure, etc. However the recursive description of the graph bracket can be used to prove the following.

**THEOREM 3.** *The graph Jones polynomial  $V_G$  is invariant under graph-theoretic Reidemeister equivalence.*

## 7. Six comments on Reidemeister equivalence

1. Diagrams of distinct classical or virtual knots may have identical looped interlacement graphs. For instance, this occurs if one diagram is obtained from another through mutation [8] or virtualization [13].

2. Some observations of Section 3 imply that several familiar features of classical knot theory do not extend to Reidemeister equivalence of graphs. For instance, a graph cannot generally be changed into a Reidemeister equivalent of an edgeless graph by adjoining or deleting loops.

3. Östlund [18] showed that the various  $\Omega.3$  moves on knot diagrams are inter-related through composition with  $\Omega.2$  moves. As illustrated in Figures 6 and 7, the same observation holds for the graph Reidemeister moves. In each of these figures, the vertical double-headed arrows represent  $\Omega.2$  moves involving the insertion or deletion of two vertices whose neighbors outside the picture are the same as those of the vertex directly above or below them. Without the dashed loop, Figure 6 illustrates the fact that the  $\Omega.3$  move pictured in Figure 5 *vi* can be obtained from the one pictured in Figure 5 *iii* through composition with  $\Omega.2$  moves; with the dashed loop it shows that the  $\Omega.3$  move pictured in Figure 5 *v* can be obtained from the one pictured in Figure 5 *i*. If all the loops in Figure 6 are toggled the resulting figure shows that part *vi* of Figure 5 can be obtained from part *ii* (without the dashed loop) and part *v* from part *iv* (with the dashed loop). Similarly, without the dashed loop Figure 7 shows that the  $\Omega.3$  move pictured in Figure 5 *ii* can be obtained by composing the one pictured in Figure 5 *vi* with  $\Omega.2$  moves; with the dashed loop Figure 7 illustrates that the move pictured in Figure 5 *iv* can be obtained from the one in Figure 5 *v*. Toggling the loops in Figure 7 we see that part *iii* of Figure 5 can be obtained from part *vi* (without the dashed loop) and part *i* from part *v* (with the dashed loop). In sum, we conclude that the  $\Omega.3$  moves pictured in Figure 5 *i*, *iv* and *v* can be obtained from each other through composition with  $\Omega.2$  moves, and the  $\Omega.3$  moves pictured in Figure 5 *ii*, *iii* and *vi* can also be obtained from each other.

4. It will come as no surprise that demonstrating the Reidemeister equivalence between two graphs can sometimes require the use of intermediate graphs which

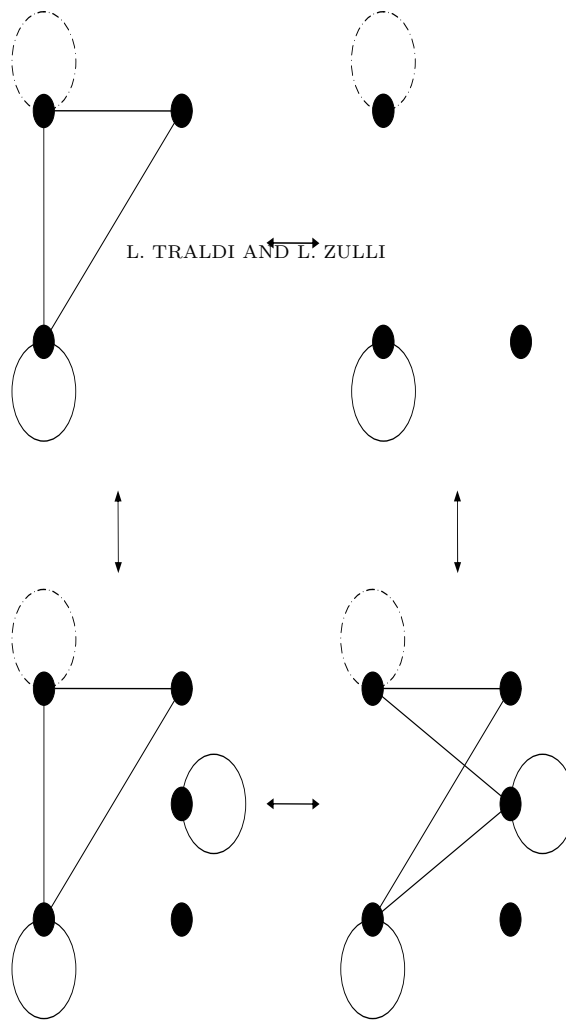


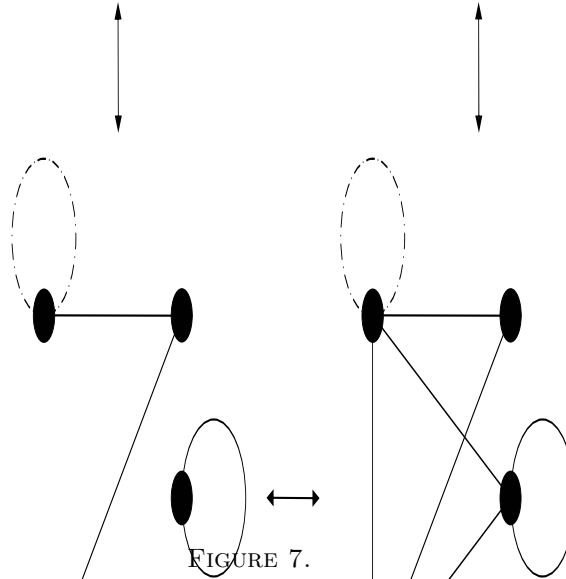
FIGURE 6.

are larger than both of the original ones. A simple example appears in Figure 8. Neither of the two graphs that appear at the top of the figure can be subjected to any Reidemeister move that does not increase the number of vertices, but as shown in the figure, the two graphs are indeed equivalent.

5. Gauss [9] observed that the double occurrence words (Gauss codes) arising from generic closed curves in the plane have the following property: An even number of other symbols appear between the two occurrences of each symbol in the word. Thus the looped interlacement graph of a classical knot diagram must be Eulerian. It follows that every Reidemeister equivalence class contains infinitely many graphs that are not looped interlacement graphs of classical knot diagrams. Let  $n \geq 1$  be an integer, let  $E_n$  be the edgeless graph with  $n$  vertices, and let  $H$  be the connected non-Eulerian graph obtained from  $E_n$  by adjoining two vertices with an  $\Omega.2$  move. (The two new vertices should be adjacent if  $n$  is even and nonadjacent if  $n$  is odd.) Given any graph  $G$ , the disjoint union  $G \cup H$  can be obtained from  $G$  with Reidemeister moves: first adjoin  $n$  isolated vertices to  $G$  using  $\Omega.1$  moves, and then use an  $\Omega.2$  move to attach the remaining two vertices of  $H$  to the first  $n$ . As  $G \cup H$  is not Eulerian, it is not the looped interlacement graph of any classical knot diagram.

6. *Circle graphs* are the intersection graphs of chord diagrams on  $\mathbb{S}^1$ ; they have received a considerable amount of attention. (See [6] for instance.) If a virtual knot diagram  $D$  is obtained from a classical knot diagram  $D'$  by designating some crossings as virtual then  $\mathcal{L}(D)$  is the subgraph of  $\mathcal{L}(D')$  induced by the vertices that





correspond to classical crossings of  $D$ , so the simple graph obtained from  $\mathcal{L}(D)$  by ignoring all loops is an induced subgraph of a circle graph. Not all simple graphs are induced subgraphs of circle graphs – for instance the wheel graph  $W_5$  is not, and hence neither is any graph which contains an induced subgraph isomorphic to  $W_5$  – and consequently, many graphs do not arise as looped interlacement graphs of virtual knot diagrams. We do not know whether or not every graph is Reidemeister equivalent to the looped interlacement graph of some virtual knot diagram.

## 8. Proof of Theorem 2

To prove Theorem 2 it is necessary to verify that whenever one applies a Reidemeister move to a (virtual) knot diagram, the effect on the looped interlacement graph is to apply one of the graph-theoretic Reidemeister moves described in Definition 7. We need not take the additional moves of virtual knot theory [16] into account, as they do not affect the looped interlacement graph.

If a new crossing is inserted using a Reidemeister move of type  $\Omega.1$ , then the new crossing is not interlaced with any other so the corresponding vertex of the looped interlacement graph is isolated.

Suppose two new crossings are inserted using a Reidemeister move of type  $\Omega.2$  as illustrated in the middle of Figure 9. We trace  $D$  by starting near the point marked  $a$  and leaving the pictured portion of  $D$ . As  $D$  is a knot diagram, we must re-enter the pictured portion first at  $b$  or  $c$ . The crossings of  $D$  that we encounter precisely once in tracing the curve from  $a$  to  $b$  or  $c$  (whichever we encounter first) are the crossings that are interlaced with the two pictured ones; consequently the vertices corresponding to the two pictured crossings have precisely

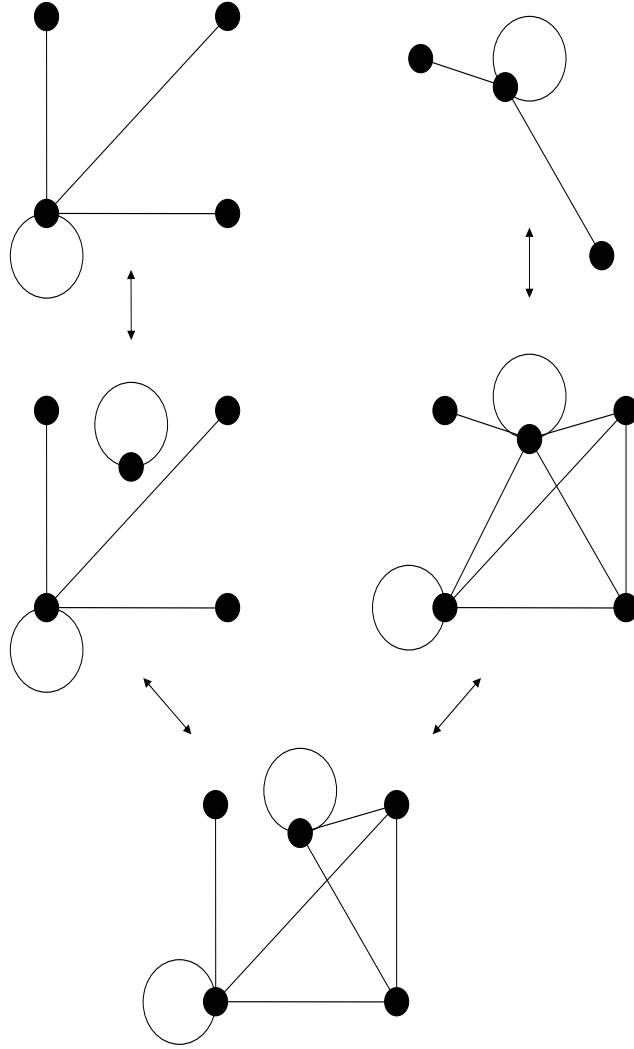


FIGURE 8. Demonstrating the Reidemeister equivalence between the two topmost graphs requires larger intermediate graphs.

the same neighbors among the other vertices. As the two pictured crossings are of opposite types, one corresponds to a looped vertex and the other corresponds to an unlooped vertex. These two vertices will be adjacent if we first re-enter the pictured portion of the diagram at  $c$ , and nonadjacent if we first re-enter the pictured portion of the diagram at  $b$ . The situation is essentially the same if the mirror-image of the Reidemeister move pictured in the middle of Figure 9 is applied.

We now discuss Reidemeister moves of type  $\Omega.3$ . By a *Reidemeister triangle* in a knot diagram we mean a three-sided complementary region with the following property: One of the three arcs that form the sides of the region passes above the other two. Call this arc  $a$ . Similarly, there is an arc  $b$  that passes below the other two, and an arc  $c$  that is “centered” between the other two. Let  $A$  denote the

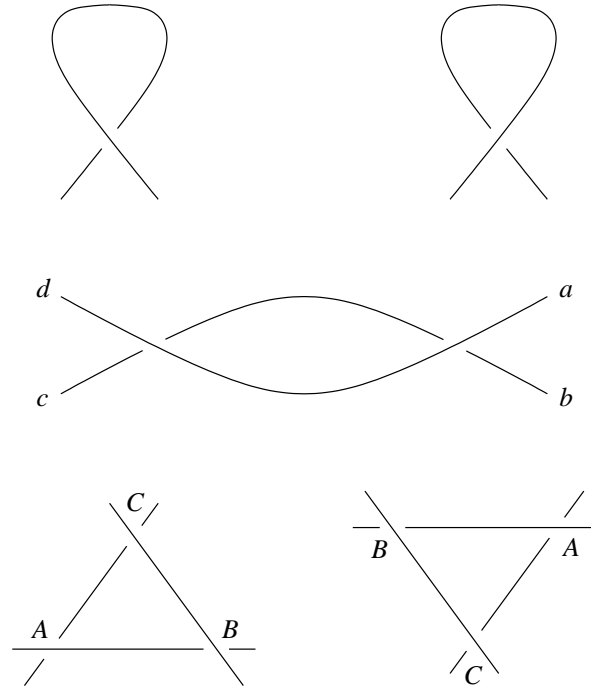


FIGURE 9. The first two types of knot-theoretic Reidemeister moves involve inserting or removing configurations like those in the top and middle rows of the figure. The third type of Reidemeister move involves “moving arc  $AB$  through crossing  $C$ ,” as indicated in the bottom row of the figure.

crossing of  $b$  and  $c$ ,  $B$  the crossing of  $a$  and  $c$ , and  $C$  the crossing of  $a$  and  $b$ . We say such a triangle is *positive* (*negative*) if  $A$ ,  $B$  and  $C$  appear in counter-clockwise (clockwise) order around the triangle. See Figure 10.

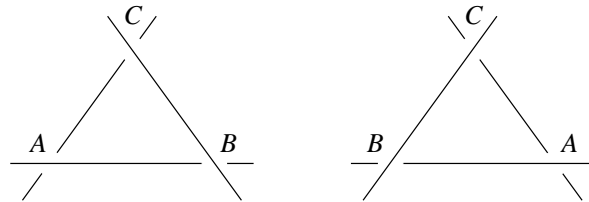


FIGURE 10. positive and negative Reidemeister triangles

If we orient the knot so that arc  $c$  is oriented from  $A$  toward  $B$ , then in the knot’s Gauss code the labels  $A$ ,  $B$  and  $C$  must appear in one of the following eight patterns:  $ABACBC$ ,  $ABACCB$ ,  $ABBCAC$ ,  $ABBCCA$ ,  $ABCABC$ ,  $ABCACB$ ,

$ABCBAC$ ,  $ABCBCA$ . Furthermore, the signs of the crossings are then determined, and hence so is the subgraph of the looped interlacement graph spanned by  $A$ ,  $B$  and  $C$ . Note that the subgraphs arising from a negative Reidemeister triangle are simply the loop-toggled versions of the subgraphs arising from a positive triangle. Also note that any crossing that is not part of the Reidemeister triangle is interlaced with either none or precisely two of  $A$ ,  $B$  and  $C$ .

Every  $\Omega.3$  move can be realized by “moving arc  $c$  through crossing  $C$ ” in a Reidemeister triangle. See Figure 9. The effect of such a move is rather minimal. A new Reidemeister triangle is created, whose arcs and crossings we label using the scheme described above. The sign of the new triangle is then the same as that of the original triangle, and the individual crossing signs of  $A$ ,  $B$  and  $C$  are unchanged. All that happens is that in each of the eight “words” above, the first and second letters are transposed, as are the third and the fourth, and the fifth and the sixth. The corresponding effect on the looped interlacement graph of the diagram is simply to toggle all the non-loop edges in the subgraph spanned by  $A$ ,  $B$  and  $C$ .

To complete the proof of Theorem 2, it suffices to verify that the six configurations depicted in Figure 5 summarize all the cases in the discussion above. The interested reader can verify the following correspondences between words and graphical configurations. (An apostrophe following a configuration indicates the toggling of all the non-loop edges in the depiction of that configuration in Figure 5.) For a positive Reidemeister triangle:  $ABACBC \leftrightarrow (iii)$ ,  $ABACCB \leftrightarrow (ii')$ ,  $ABBCAC \leftrightarrow (iii')$ ,  $ABBCCA \leftrightarrow (v')$ ,  $ABCABC \leftrightarrow (v)$ ,  $ABCACB \leftrightarrow (iii)$ ,  $ABCBAC \leftrightarrow (ii)$ ,  $ABCBCA \leftrightarrow (iii')$ . For a negative Reidemeister triangle:  $ABACBC \leftrightarrow (iv)$ ,  $ABACCB \leftrightarrow (i')$ ,  $ABBCAC \leftrightarrow (iv')$ ,  $ABBCCA \leftrightarrow (vi')$ ,  $ABCABC \leftrightarrow (vi)$ ,  $ABCACB \leftrightarrow (iv)$ ,  $ABCBAC \leftrightarrow (i)$ ,  $ABCBCA \leftrightarrow (iv')$ .

### 9. Proof of Theorem 3

The definition of the graph Jones polynomial can be “explained” in much the same way Kauffman explained the definition of the knot Jones polynomial in [15]. Suppose we seek an evaluation of the graph bracket that is invariant under the second and third types of Reidemeister moves. Consider  $G_1$  and  $G_2$  with  $V(G_1) = \{u, v, w\} = V(G_2)$ ,  $E(G_1) = \{\{u, v\}, \{v, w\}\}$  and  $E(G_2) = \{\{u, w\}, \{v, w\}, \{v, v\}\}$ ; they have  $[G_1] = [G_2] + (d - 1)A(B^2 + ABd + A^2)$ . If an evaluation in an integral domain is to yield equal values for  $[G_1]$  and  $[G_2]$  then one of  $d - 1$ ,  $A$ ,  $B^2 + ABd + A^2$  should evaluate to 0; the only option that could possibly yield an interesting invariant is  $B^2 + ABd + A^2 \mapsto 0$ . Now consider  $H_1$  and  $H_2$  with  $V(H_1) = \emptyset = E(H_1)$ ,  $V(H_2) = \{v, w\}$  and  $E(H_2) = \{\{v, v\}\}$ . They have  $[H_1] = 1$  and  $[H_2] = AB + d(B^2 + ABd + A^2)$ ; if  $B^2 + ABd + A^2 \mapsto 0$  then we are naturally led to the evaluations  $AB \mapsto 1$  and (hence)  $d \mapsto -A^2 - A^{-2}$ . Finally, the factor  $(-1)^n \cdot t^{(3n-6\ell)/4}$  is introduced to cancel variation under Reidemeister moves of the first type, and to maintain invariance under moves of the second and third types. Of course this “explanation” only rationalizes the definition of the graph Jones polynomial; it does not actually prove that  $V_G$  is invariant under graph Reidemeister moves.

The invariance of  $V_G$  under  $\Omega.1$  moves follows directly from Proposition 1, because the two one-vertex graphs both have graph Jones polynomial 1. The recursive description of the graph Jones polynomial that results from Theorem 1 is

complicated by the fact that the appropriate coefficients  $t^{(3n-6\ell)/4}$  for the Jones polynomials of  $G^a$ ,  $G^{ab}$  and  $(G^{ab})^a$  will vary according to the configuration of looped vertices in  $G$ . Consequently the rest of the proof of Theorem 3 is focused on the reduced graph bracket rather than the Jones polynomial itself.

**PROPOSITION 5.** *Suppose  $G$  is obtained from the edgeless graph  $H = E_n$  by adjoining two vertices  $v$  and  $w$  in an  $\Omega.2$  move. Then  $\langle G \rangle = \langle H \rangle = (-A^3)^n$  and  $V_G = V_H = 1$ .*

**PROOF.** If  $n = 0$  then  $V(G) = \{v, w\}$  and  $E(G)$  is  $\{\{v, v\}\}$  or  $\{\{v, v\}, \{v, w\}\}$ . Then  $[G]$  is  $A^2d + AB + ABd^2 + B^2d$  or  $A^2 + ABd + AB + B^2$  (respectively), so  $\langle G \rangle$  is  $-A^4 - 1 + 1 + A^4 + 2 + A^{-4} - A^{-4} - 1 = 1$  or  $A^2 - A^{-2} - A^2 + 1 + A^{-2} = 1$ ; in either case  $V_G = (-1)^2 \cdot t^{(3(2)-6(1))/4} \cdot \langle G \rangle = 1$ .

Proceeding inductively, suppose  $n > 0$ . If  $u$  is an isolated vertex of  $G$  then  $V_G = V_{G-u}$  and the inductive hypothesis implies  $V_{G-u} = 1$ . Hence we may as well assume that  $G$  has no isolated vertex, i.e., that all  $n$  vertices of  $H$  are adjacent to both  $v$  and  $w$ . Then all the vertices of  $G^{h_n w} - h_n - w$  and  $(G^{h_n w})^{h_n} - h_n - w$  are isolated. As  $G^{h_n w} - h_n - w$  has one loop and  $(G^{h_n w})^{h_n} - h_n - w$  has no loops, it follows that  $\langle G^{h_n w} - h_n - w \rangle = (-1)^n A^{3n-6}$  and  $\langle (G^{h_n w})^{h_n} - h_n - w \rangle = (-1)^n A^{3n}$ . In addition,  $G^{h_n} - h_n$  differs from  $G - h_n$  only in that the loop at  $v$  has been moved to  $w$  and the adjacency of  $v$  and  $w$  has been toggled, so the inductive hypothesis implies that  $G^{h_n} - h_n$  has Jones polynomial 1 and  $\langle G^{h_n} - h_n \rangle = (-A^3)^{n-1}$ .

Part (ii) of the recursive description of the graph bracket tells us that

$$\begin{aligned} \langle G \rangle &= A^2 \langle G^{h_n w} - h_n - w \rangle + \langle (G^{h_n w})^{h_n} - h_n - w \rangle + A^{-1} \langle G^{h_n} - h_n \rangle \\ &= (-1)^n A^{3n-4} + (-1)^n A^{3n} + A^{-1} (-A^3)^{n-1} = (-1)^n A^{3n}. \end{aligned}$$

As  $G$  has  $n + 2$  vertices and one loop, it follows that  $V_G = 1$ .  $\square$

**PROPOSITION 6.** *Suppose  $G$  is obtained from  $H$  by adjoining two vertices  $v$  and  $w$  in an  $\Omega.2$  move. Then  $\langle G \rangle = \langle H \rangle$  and  $V_G = V_H$ .*

**PROOF.** If  $H$  has no edges then Proposition 5 applies. The proof proceeds by induction on  $n = |V(H)|$  and for each value of  $n$ , by induction on the number of loops in  $H$ .

Suppose  $H$  has a loop at a vertex  $a$ . Then  $G - \{a, a\}$  has fewer loops than  $G$  has, and  $G^a - a$  has fewer vertices. Moreover,  $G - \{a, a\}$  and  $G^a - a$  are obtained from  $H - \{a, a\}$  and  $H^a - a$  (respectively) by Reidemeister moves of type  $\Omega.2$ , so we may inductively assume that  $\langle G - \{a, a\} \rangle = \langle H - \{a, a\} \rangle$  and  $\langle G^a - a \rangle = \langle H^a - a \rangle$ . Then  $\langle G \rangle = A^{-2} \langle G - \{a, a\} \rangle + (A - A^{-3}) \langle G^a - a \rangle = A^{-2} \langle H - \{a, a\} \rangle + (A - A^{-3}) \langle H^a - a \rangle = \langle H \rangle$ ; as  $G$  has two more vertices and one more loop than  $H$ , this implies  $V_G = V_H$ .

Suppose now that  $H$  has no loops and  $a, b \in V(H)$  are adjacent. Then  $G^{ab} - a - b$ ,  $(G^{ab})^a - a - b$  and  $G^a - a$  are obtained from  $H^{ab} - a - b$ ,  $(H^{ab})^a - a - b$  and  $H^a - a$  (respectively) by Reidemeister moves of type  $\Omega.2$ , so the inductive hypothesis tells us that  $\langle G^{ab} - a - b \rangle = \langle H^{ab} - a - b \rangle$ ,  $\langle (G^{ab})^a - a - b \rangle = \langle (H^{ab})^a - a - b \rangle$  and  $\langle G^a - a \rangle = \langle H^a - a \rangle$ . Then  $\langle G \rangle = A^2 \langle G^{ab} - a - b \rangle + \langle (G^{ab})^a - a - b \rangle + A^{-1} \langle G^a - a \rangle = A^2 \langle H^{ab} - a - b \rangle + \langle (H^{ab})^a - a - b \rangle + A^{-1} \langle H^a - a \rangle = \langle H \rangle$ . As  $G$  has one loop and  $H$  has none, this implies  $V_G = V_H$ .  $\square$

LEMMA 1. *Suppose  $m \geq 1$ . Let  $\Gamma$  have  $V(\Gamma) = \{v_1, \dots, v_{m+1}\}$  and  $E(\Gamma) = \{v_1, v_2\} \cup \{\{v_i, v_1\}, \{v_i, v_2\} \mid 3 \leq i \leq m+1\}$ , and let  $\Gamma'$  have  $V(\Gamma') = \{v_1, \dots, v_{m+2}\}$  and  $E(\Gamma') = \{\{v_i, v_1\}, \{v_i, v_2\} \mid 3 \leq i \leq m+2\}$ . Then  $V_\Gamma = V_{\Gamma'}$ .*

PROOF.  $(\Gamma')^{v_{m+2}v_1}$  is obtained from  $\Gamma'$  by toggling away all the adjacencies between  $v_2$  and the  $v_i$ ,  $2 < i < m+2$ ; hence  $(\Gamma')^{v_{m+2}v_1} - v_1 - v_{m+2} = E_m$  and  $((\Gamma')^{v_{m+2}v_1})^{v_{m+2}} - v_1 - v_{m+2} = E_{m-1} \cup L_1$ , where  $L_1$  consists of a single looped vertex.  $(\Gamma')^{v_{m+2}} - v_{m+2}$  is obtained from  $\Gamma' - v_{m+2}$  by adjoining an edge  $\{v_1, v_2\}$  and also adjoining loops at  $v_1$  and  $v_2$ . Hence  $((\Gamma')^{v_{m+2}} - v_{m+2}) - \{v_1, v_1\}$  is obtained from  $E_{m-1}$  by adjoining  $v_1$  and  $v_2$  in an  $\Omega.2$  move, and  $((\Gamma')^{v_{m+2}} - v_{m+2})^{v_1} - v_1$  is the graph in which  $v_2$  is unlooped and isolated and  $v_3, \dots, v_{m+1}$  are all looped and all adjacent to each other; that is,  $((\Gamma')^{v_{m+2}} - v_{m+2})^{v_1} - v_1$  is the disjoint union of an isomorphic copy of  $E_1$  and an isomorphic copy of  $K_{m-1} + I$ , the complete looped graph with  $m-1$  vertices.

We conclude that

$$\begin{aligned} \langle \Gamma' \rangle &= A^2 \langle (\Gamma')^{v_{m+2}v_1} - v_1 - v_{m+2} \rangle + \langle ((\Gamma')^{v_{m+2}v_1})^{v_{m+2}} - v_1 - v_{m+2} \rangle \\ &\quad + A^{-1} \langle (\Gamma')^{v_{m+2}} - v_{m+2} \rangle \\ &= A^2(-A^3)^m + (-A^3)^{m-1}(-A^{-3}) + A^{-3} \langle ((\Gamma')^{v_{m+2}} - v_{m+2}) - \{v_1, v_1\} \rangle \\ &\quad + A^{-1}(A - A^{-3}) \langle ((\Gamma')^{v_{m+2}} - v_{m+2})^{v_1} - v_1 \rangle \\ &= A^2(-A^3)^m + (-A^3)^{m-1}(-A^{-3}) + A^{-3}(-A^3)^{m-1} \\ &\quad + A^{-1}(A - A^{-3})(-A^3) \langle K_{m-1} + I \rangle \\ &= (-1)^m A^{3m+2} + A^{-1}(A - A^{-3})(-A^3) \langle K_{m-1} + I \rangle. \end{aligned}$$

$\Gamma^{v_{m+1}v_1}$  is obtained from  $\Gamma$  by toggling away all the adjacencies between  $v_2$  and the  $v_i$ ,  $3 \leq i \leq m$ ; hence  $\Gamma^{v_{m+1}v_1} - v_{m+1} - v_1 = E_{m-1}$  and  $(\Gamma^{v_{m+1}v_1})^{v_{m+1}} - v_{m+1} - v_1 = E_{m-2} \cup L_1$ .  $\Gamma^{v_{m+1}} - v_{m+1}$  is obtained from  $\Gamma - v_{m+1}$  by removing the edge  $\{v_1, v_2\}$  and adjoining loops at  $v_1$  and  $v_2$ . Consequently  $(\Gamma^{v_{m+1}} - v_{m+1}) - \{v_1, v_1\}$  is obtained from  $E_{m-2}$  by adjoining  $v_1$  and  $v_2$  in an  $\Omega.2$  move, and  $(\Gamma^{v_{m+1}} - v_{m+1})^{v_1} - v_1$  is the graph in which  $v_2, \dots, v_m$  are all looped and all adjacent to each other; that is,  $(\Gamma^{v_{m+1}} - v_{m+1})^{v_1} - v_1$  is an isomorphic copy of  $K_{m-1} + I$ .

We conclude that

$$\begin{aligned} \langle \Gamma \rangle &= A^2 \langle \Gamma^{v_{m+1}v_1} - v_1 - v_{m+1} \rangle + \langle (\Gamma^{v_{m+1}v_1})^{v_{m+1}} - v_{m+1} - v_1 \rangle \\ &\quad + A^{-1} \langle \Gamma^{v_{m+1}} - v_{m+1} \rangle \\ &= A^2(-A^3)^{m-1} + (-A^3)^{m-2}(-A^{-3}) + A^{-3} \langle (\Gamma^{v_{m+1}} - v_{m+1}) - \{v_1, v_1\} \rangle \\ &\quad + A^{-1}(A - A^{-3}) \langle (\Gamma^{v_{m+1}} - v_{m+1})^{v_1} - v_1 \rangle \\ &= A^2(-A^3)^{m-1} + (-A^3)^{m-2}(-A^{-3}) + A^{-3}(-A^3)^{m-2} \\ &\quad + A^{-1}(A - A^{-3}) \langle K_{m-1} + I \rangle \\ &= (-1)^{m-1} A^{3m-1} + A^{-1}(A - A^{-3}) \langle K_{m-1} + I \rangle, \end{aligned}$$

and hence  $\langle \Gamma' \rangle = -A^3 \cdot \langle \Gamma \rangle$ . As  $\Gamma'$  has one more vertex than  $\Gamma$  and neither has any loops, this implies  $V_{\Gamma'} = V_\Gamma$ .  $\square$

PROPOSITION 7. *Suppose  $G$  has three vertices  $u, v, w$  which span a subgraph  $H$  isomorphic to one of those pictured in Figure 5; suppose further that every vertex outside  $H$  is unlooped, has degree 2 and is adjacent to two of  $u, v, w$ . Let  $G'$  be*

the graph obtained from  $G$  by toggling all the non-loop adjacencies in  $H$ . Then  $\langle G \rangle = \langle G' \rangle$  and hence  $V_G = V_{G'}$ .

PROOF. In case  $G = H$  the result can be verified by direct computation, which we leave to the reader. (It is interesting to note that even in this simple case, the proposition fails for the 3-vertex configurations not pictured in Figure 5.)

We proceed inductively, assuming that  $G$  has  $n \geq 4$  vertices. Let  $V(G) - V(H) = N_{uv} \cup N_{uw} \cup N_{vw}$ , with the elements of  $N_{ij}$  adjacent to  $i$  and  $j$ .

Case 1. Suppose  $H$  is isomorphic to the graph appearing in Figure 5 *vi*, with a loop at  $u$ .

Suppose  $a \in N_{uv}$ . Then  $G^{av}$  is obtained from  $G$  by toggling all adjacencies between  $u$  and neighbors of  $v$  other than  $a$  and  $u$ ; hence  $V(G^{av} - a - v) = \{u, w\} \cup (N_{uv} - \{a\}) \cup N_{uw} \cup N_{vw}$  and  $E(G^{av} - a - v) = E(G - a - v) \cup \{\{u, y\} | y \in N_{vw}\} - \{\{u, y\} | y \in N_{uv}\} - \{u, w\}$ . Similarly,  $(G')^{av}$  is obtained from  $G'$  by toggling all adjacencies between  $u$  and neighbors of  $v$  other than  $a$ ; hence  $V((G')^{av} - a - v) = \{u, w\} \cup (N_{uv} - \{a\}) \cup N_{uw} \cup N_{vw}$  and  $E((G')^{av} - a - v) = E(G' - a - v) \cup \{\{u, y\} | y \in N_{vw}\} - \{\{u, y\} | y \in N_{uv}\}$ . We see that  $G^{av} - a - v = (G')^{av} - a - v$ . The only differences between  $G^{av}$  and  $(G^{av})^a$  are that the latter has a loop at  $v$  instead of  $u$  and no edge  $\{u, v\}$ ; consequently  $(G^{av})^a - a - v$  coincides with  $G^{av} - a - v$  except for the fact that  $(G^{av})^a - a - v$  has no loop at  $u$ . Similarly, the only difference between  $(G')^{av} - a - v$  and  $((G')^{av})^a - a - v$  is that the latter has no loop at  $u$ ; hence  $(G^{av})^a - a - v = ((G')^{av})^a - a - v$ .

$G^a$  is obtained from  $G$  by moving the loop from  $u$  to  $v$  and removing the edge  $uv$ , and  $(G')^a$  is obtained from  $G'$  by moving the loop from  $u$  to  $v$  and adjoining an edge  $\{u, v\}$ . Consequently the full subgraph of  $G^a - a$  with vertices  $u, v, w$  is isomorphic to the graph pictured in Figure 5 *iv*, and  $(G')^a - a$  is obtained from  $G^a - a$  by performing a Reidemeister move of type  $\Omega.3$  on this subgraph. The inductive hypothesis tells us that  $V_{G^a - a} = V_{(G')^a - a}$ ; as  $G^a - a$  and  $(G')^a - a$  both have one loop and  $n - 1$  vertices, this implies that  $\langle G^a - a \rangle = \langle (G')^a - a \rangle$ . We conclude that

$$\begin{aligned} \langle G \rangle &= A^2 \langle G^{av} - a - v \rangle + \langle (G^{av})^a - a - v \rangle + A^{-1} \langle G^a - a \rangle \\ &= A^2 \langle (G')^{av} - a - v \rangle + \langle ((G')^{av})^a - a - v \rangle + A^{-1} \langle (G')^a - a \rangle = \langle G' \rangle. \end{aligned}$$

If there is an  $a \in N_{uw}$  the same argument applies, with  $v$  and  $w$  interchanged throughout.

Suppose  $N_{uv} = \emptyset = N_{uw}$ , so that  $V(G) = \{u, v, w\} \cup N_{vw}$ ; let  $a \in N_{vw}$  and let  $m = n - 3$ . Then  $G$  is obtained from the graph denoted  $\Gamma$  in Lemma 1 by adjoining  $u$  and  $a$  through an  $\Omega.2$  move, and  $G'$  is obtained from the graph denoted  $\Gamma'$  in Lemma 1 by adjoining the isolated, looped vertex  $u$ . It follows that  $V_G = V_\Gamma = V_{\Gamma'} = V_{G'}$ .

Case 2. Suppose  $H$  is isomorphic to the graph appearing in Figure 5 *ii*, with  $v$  adjacent to both  $u$  and  $w$ .

Suppose  $a \in N_{uv}$ . Then  $G^{av}$  is obtained from  $G$  by toggling all adjacencies between  $u$  and neighbors of  $v$  other than  $a$  and  $u$ ; hence  $V(G^{av} - a - v) = \{u, w\} \cup (N_{uv} - \{a\}) \cup N_{uw} \cup N_{vw}$  and  $E(G^{av} - a - v) = E(G - a - v) \cup \{\{u, w\}\} \cup \{\{u, y\} | y \in N_{vw}\} - \{\{u, y\} | y \in N_{uv}\}$ . Similarly,  $(G')^{av}$  is obtained from  $G'$  by toggling all adjacencies between  $u$  and neighbors of  $v$  other than  $a$ ; hence  $V((G')^{av} - a - v) = \{u, w\} \cup (N_{uv} - \{a\}) \cup N_{uw} \cup N_{vw}$  and  $E((G')^{av} - a - v) = E(G' - a - v) \cup \{\{u, y\} | y \in N_{vw}\} - \{\{u, y\} | y \in N_{uv}\}$ . We see that  $G^{av} - a - v = (G')^{av} - a - v$ . The only differences between  $G^{av}$  and  $(G^{av})^a$  are that the latter has loops at  $u$  and  $v$  and

no edge  $\{u, v\}$ ; consequently  $(G^{av})^a - a - v$  coincides with  $G^{av} - a - v$  except for the fact that  $(G^{av})^a - a - v$  has a loop at  $u$ . Similarly, the only difference between  $(G')^{av} - a - v$  and  $((G')^{av})^a - a - v$  is that the latter has a loop at  $u$ ; hence  $(G^{av})^a - a - v = ((G')^{av})^a - a - v$ .

$G^a$  is obtained from  $G$  by adjoining loops at  $u$  and  $v$  and removing the edge  $\{u, v\}$ , and  $(G')^a$  is obtained from  $G'$  by adjoining loops at  $u$  and  $v$  and also adjoining an edge  $\{u, v\}$ . Consequently the full subgraph of  $(G')^a - a$  with vertices  $u, v, w$  is isomorphic to the graph pictured in Figure 5 *iii*, and  $G^a - a$  is obtained from  $(G')^a - a$  by performing an  $\Omega.3$  move on this subgraph. The inductive hypothesis tells us that  $V_{G^a - a} = V_{(G')^a - a}$ ; as  $G^a - a$  and  $(G')^a - a$  both have two loops and  $n - 1$  vertices, this implies that  $\langle G^a - a \rangle = \langle (G')^a - a \rangle$ . We conclude that

$$\begin{aligned} \langle G \rangle &= A^2 \langle G^{av} - a - v \rangle + \langle (G^{av})^a - a - v \rangle + A^{-1} \langle G^a - a \rangle \\ &= A^2 \langle (G')^{av} - a - v \rangle + \langle ((G')^{av})^a - a - v \rangle + A^{-1} \langle (G')^a - a \rangle = \langle G' \rangle. \end{aligned}$$

If there is an  $a \in N_{vw}$  the same argument applies, with  $u$  and  $w$  interchanged throughout.

Suppose  $N_{uv} = \emptyset = N_{vw}$ , so that  $V(G) = \{u, v, w\} \cup N_{uw}$ ; let  $m = n - 2$ . Then  $G$  is the graph denoted  $\Gamma'$  in Lemma 1, and  $G'$  is obtained from the graph denoted  $\Gamma$  in Lemma 1 by adjoining the isolated, unlooped vertex  $v$ . It follows that  $V_G = V_{\Gamma'} = V_{\Gamma} = V_{G'}$ .

Case 3. Suppose  $H$  is isomorphic to the graph appearing in Figure 5 *iv*, with a loop at  $u$  and  $v$  not adjacent to  $u$ . Then  $G - \{u, u\}$  and  $G' - \{u, u\}$  are related as in Case 2, so  $\langle G - \{u, u\} \rangle = \langle G' - \{u, u\} \rangle$ .

Observe that  $G^u - u$  and  $(G')^u - u$  share the same subgraph  $S$  spanned by  $N_{uv} \cup N_{uw} \cup N_{vw}$ : loops and non-loop adjacencies within  $N_{uv} \cup N_{uw}$  are toggled from those of  $G$ , and loops and adjacencies involving elements of  $N_{vw}$  are the same as those of  $G$ . In  $G^u - u$ , all elements of  $N_{uv} \cup N_{vw}$  are adjacent to both  $v$  and  $w$ , and no element of  $N_{uw}$  is adjacent to either  $v$  or  $w$ .  $G^u - u$  has a loop at  $w$  and no loop at  $v$ , so  $G^u - u$  is obtained from  $S$  by adjoining  $v$  and  $w$  in an  $\Omega.2$  move. In  $(G')^u - u$ , all elements of  $N_{uv} \cup N_{vw}$  are adjacent to both  $v$  and  $w$ , and no element of  $N_{uw}$  is adjacent to either  $v$  or  $w$ .  $(G')^u - u$  has a loop at  $v$  and no loop at  $w$ , so  $(G')^u - u$  is obtained from  $S$  by adjoining  $v$  and  $w$  in an  $\Omega.2$  move. It follows that  $\langle G^u - u \rangle = \langle (G')^u - u \rangle$ .

We conclude that

$$\begin{aligned} \langle G \rangle &= A^{-2} \langle G - \{u, u\} \rangle + (A - A^{-3}) \langle G^u - u \rangle \\ &= A^{-2} \langle G' - \{u, u\} \rangle + (A - A^{-3}) \langle (G')^u - u \rangle = \langle G' \rangle. \end{aligned}$$

As the graphs appearing on the left-hand side of Figure 5 result from toggling the loops in the graphs that appear on the right-hand side, the remaining three cases follows from the first three and Proposition 2.  $\square$

**PROPOSITION 8.** *Suppose  $G$  has three vertices  $u, v, w$  which span a subgraph  $H$  isomorphic to one of those pictured in Figure 5; suppose further that every vertex outside  $H$  that is adjacent to any of  $u, v, w$  is adjacent to precisely two of  $u, v, w$ . If  $G'$  is the graph obtained from  $G$  by toggling all the non-loop adjacencies in  $H$  then  $\langle G \rangle = \langle G' \rangle$  and  $V_G = V_{G'}$ .*

**PROOF.** If  $G$  has no vertices outside  $H$  then we appeal to Proposition 7; the proof proceeds by induction on  $|V(G)| = n \geq 4$ .



If  $|V(G)| = n$  and every edge of  $G$  is incident on  $H$  then we appeal to Proposition 7 (and Proposition 1, if necessary); we proceed by induction on the number of edges of  $G$  not incident on  $H$ .

Suppose  $G$  has a looped vertex  $a \notin V(H)$ . The inductive hypothesis tells us that  $\langle G - \{a, a\} \rangle = \langle G' - \{a, a\} \rangle$ . If  $a$  is not adjacent to any of  $u, v, w$  then local complementation at  $a$  does not affect any edges incident on  $H$ , so  $G^a - a$  and  $(G')^a - a$  are related through a type 3 Reidemeister move represented by the same part of Figure 5 as  $G$  and  $G'$ . Suppose  $a$  is adjacent to precisely two of  $u, v, w$ . If  $b \in V(G) - \{u, v, w, a\}$  is not adjacent to  $a$ , then local complementation at  $a$  does not affect adjacencies between  $b$  and  $u, v, w$ . If  $b \in V(G) - \{u, v, w, a\}$  is adjacent to  $a$ , then local complementation at  $a$  toggles two of the adjacencies between  $b$  and  $u, v, w$ . In any case, the fact that  $b$  is adjacent to an even number of  $u, v, w$  in  $G$  implies that  $b$  is also adjacent to an even number of  $u, v, w$  in  $G^a$ . Local complementation at  $a$  transforms  $H$  into another one of the three-vertex configurations of Figure 5; for instance, if  $H$  is isomorphic to Figure 5 *iii* and  $a$  is adjacent to the two looped vertices then the subgraph of  $(G')^a - a$  spanned by  $u, v, w$  is isomorphic to Figure 5 *ii*. We conclude that  $G^a - a$  and  $(G')^a - a$  are related through an  $\Omega.3$  move; as both have only  $|V(G)| - 1$  vertices, we may assume inductively that  $\langle G^a - a \rangle = \langle (G')^a - a \rangle$  and hence

$$\begin{aligned} \langle G \rangle &= A^{-2} \langle G - \{a, a\} \rangle + (A - A^{-3}) \langle G^a - a \rangle \\ &= A^{-2} \langle G' - \{a, a\} \rangle + (A - A^{-3}) \langle (G')^a - a \rangle = \langle G' \rangle. \end{aligned}$$

Suppose  $G$  has no loops but has an edge  $\{a, b\}$  not incident on  $H$ . As in the preceding paragraph,  $G^a - a$  and  $(G')^a - a$  are also related through a Reidemeister move of type 3 so we may assume inductively that  $\langle G^a - a \rangle = \langle (G')^a - a \rangle$ . If either  $a$  or  $b$  is adjacent to none of  $u, v, w$  then passing from  $G$  to  $G^{ab}$  does not affect  $H$ ; the same is true if  $a$  and  $b$  are adjacent to the same two of  $u, v, w$ . Suppose  $a$  and  $b$  are adjacent to different pairs of vertices of  $H$ . Then one of  $u, v, w$  is adjacent to  $a$  and  $b$ , one is adjacent to  $a$  and not  $b$ , and one is adjacent to  $b$  and not  $a$ . If  $x \in V(G) - \{a, b, u, v, w\}$  is adjacent to neither  $a$  nor  $b$  then passing from  $G$  to  $G^{ab}$  does not affect any adjacencies between  $x$  and  $u, v, w$ ; if  $x$  is adjacent to  $a$  or  $b$  then passing from  $G$  to  $G^{ab}$  toggles precisely two adjacencies between  $x$  and  $u, v, w$  (the two whose adjacencies with  $a, b$  do not match those of  $x$ ). Consequently the fact that  $x$  is adjacent to an even number of  $u, v, w$  in  $G$  implies that  $x$  is also adjacent to an even number of  $u, v, w$  in  $G^{ab}$ . As all three non-loop adjacencies involving  $u, v, w$  are toggled in passing from  $G$  to  $G^{ab}$ , we conclude that  $(G')^{ab}$  and  $G^{ab}$  are related through the same  $\Omega.3$  Reidemeister move as  $G$  and  $G'$ , respectively. As in the preceding paragraph, it follows that  $(G^{ab})^a - a$  and  $((G')^{ab})^a - a$  are also related through  $\Omega.3$  moves. We conclude inductively that

$$\begin{aligned} \langle G \rangle &= A^2 \langle G^{ab} - a - b \rangle + \langle (G^{ab})^a - a - b \rangle + A^{-1} \langle G^a - a \rangle \\ &= A^2 \langle (G')^{ab} - a - b \rangle + \langle ((G')^{ab})^a - a - b \rangle + A^{-1} \langle (G')^a - a \rangle = \langle G' \rangle. \end{aligned}$$

□

### Acknowledgments

We thank the anonymous referee for several comments and corrections that significantly improved the paper. We are also grateful to Lafayette College for supporting our work.

## References

- [1] R. Arratia, B. Bollobás, and G. B. Sorkin, The interlace polynomial: A new graph polynomial, in: *Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms* (San Francisco, 2000), Association for Computing Machinery, New York, 2000, ps. 237-245.
- [2] R. Arratia, B. Bollobás, and G. B. Sorkin, *The interlace polynomial of a graph*, J. Combin. Theory Ser. B **92** (2004), 199-233.
- [3] R. Arratia, B. Bollobás, and G. B. Sorkin, *A two-variable interlace polynomial*, Combinatorica **24** (2004), 567-584.
- [4] P. N. Balister, B. Bollobás, O. M. Riordan and A. D. Scott, *Alternating knot diagrams, Euler circuits and the interlace polynomial*, Europ. J. Combinatorics **22** (2001), 1-4.
- [5] J. S. Birman and T. Kanenobu, *Jones' braid-plat formula and a new surgery triple*, Proc. Amer. Math. Soc. **102** (1988), 687-695.
- [6] A. Bouchet, *Circle graph obstructions*, J. Combin. Theory Ser. B **60** (1994), 107-144.
- [7] A. Bouchet, *Multimatroids III. Tightness and fundamental graphs*, Europ. J. Combinatorics **22** (2001) 657-677.
- [8] S. V. Chmutov and S. K. Lando, *Mutant knots and intersection graphs*, Algebr. Geom. Topol. **7** (2007), 1579-1598.
- [9] C. F. Gauss, *Werke*, Band VIII, Teubner, Leipzig, 1900.
- [10] S. Huggett, *On tangles and matroids*, J. Knot Theory Ramifications **14** (2005), 919-929.
- [11] F. Jaeger, D. L. Vertigan and D. J. A. Welsh, *On the computational complexity of the Jones and Tutte polynomials*, Math. Proc. Cambridge Philos. Soc. **108** (1990), 35-53.
- [12] V. F. R. Jones, *A polynomial invariant for links via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), 103-112.
- [13] L. H. Kauffman, *Detecting virtual knots*, Atti Semin. Mat. Fis. Univ. Modena **49** (2001), Suppl., 241-282.
- [14] L. H. Kauffman, *Knot diagrammatics*, in: Handbook of Knot Theory, W. Menasco and M. Thistlethwaite, eds., Elsevier B. V., Amsterdam, 2005.
- [15] L. H. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), 395-407.
- [16] L. H. Kauffman, *Virtual knot theory*, Europ. J. Combinatorics **20** (1999), 663-691.
- [17] J. Lannes, *Sur l'invariant de Kervaire des noeuds classiques*, Comment. Math. Helv. **60** (1985), 179-192.
- [18] O.-P. Östlund, *Invariants of knot diagrams and relations among Reidemeister moves*, J. Knot Theory Ramifications **10** (2001), 1215-1227.
- [19] K. Reidemeister, *Knotentheorie*, Springer, Berlin, 1932.
- [20] B. Shtylla and L. Zulli, *An extension of the Jones polynomial of classical knots*, J. Knot Theory Ramifications **15** (2006), 81-100.
- [21] M. B. Thistlethwaite, *A spanning tree expansion of the Jones polynomial*, Topology **26** (1987), 297-309.
- [22] L. Zulli, *A matrix for computing the Jones polynomial of a knot*, Topology **34** (1995), 717-729.

LAFAYETTE COLLEGE, EASTON, PENNSYLVANIA 18042

E-mail address: traldil@lafayette.edu

URL: <http://www.lafayette.edu/~traldil/>

LAFAYETTE COLLEGE, EASTON, PENNSYLVANIA 18042

E-mail address: zullil@lafayette.edu

URL: <http://www.cs.lafayette.edu/~zullil/>